Supplementary Material

A Proof of Theorem 1

Proof. We rewrite the objective function as:

$$
\begin{aligned}
&F(\tilde{\mathbf{A}}) = m\sum_{p,q=1}^{n}\tilde{A}_{pq}\log\tilde{A}_{pq} - \sum_{p,q=1}^{n}\tilde{A}_{pq}\sum_{i=1}^{m}\log(A_{pq}^{(i)} - E_{pq}^{(i)}) \\
&+ m\sum_{p,q=1}^{n}(1-\tilde{A}_{pq})\log(1-\tilde{A}_{pq}) - \sum_{p,q=1}^{n}(1-\tilde{A}_{pq})\sum_{i=1}^{m}\log(1-A_{pq}^{(i)} + E_{pq}^{(i)}) \\
&+ \lambda_2 tr(\tilde{\mathbf{A}}^T\mathbf{S}^{-1}\tilde{\mathbf{A}}) \\
&= m\sum_{p,q=1}^{n}\tilde{A}_{pq}\log\tilde{A}_{pq} - \sum_{p,q=1}^{n}\tilde{A}_{pq}\sum_{i=1}^{m}\log\frac{A_{pq}^{(i)} - E_{pq}^{(i)}}{1-A_{pq}^{(i)} + E_{pq}^{(i)}} \\
&+ m\sum_{p,q=1}^{n}(1-\tilde{A}_{pq})\log(1-\tilde{A}_{pq}) + \lambda_2\mathrm{tr}(\tilde{\mathbf{A}}^T\mathbf{S}^{-1}\tilde{\mathbf{A}}) + \mathcal{N} \qquad (1)
\end{aligned}
$$

where $\mathcal{N} = -\sum_{p,q=1}^{n}\sum_{i=1}^{m}\log(1-A_{pq}^{(i)} + E_{pq}^{(i)})$ is constant term which does not contain $\tilde{\mathbf{A}}$, thus we can drop $\mathcal{N}$ safely. Introducing C_{pq}^{+}, C_{pq}^{-}, D_{pq}^{+}, and D_{pq}^{-} which are defined in section 3.2, we get:

$$
\begin{aligned}
&F(\tilde{\mathbf{A}}) = m\sum_{p,q=1}^{n}\tilde{A}_{pq}\log\tilde{A}_{pq} - \sum_{p,q=1}^{n}\tilde{A}_{pq}C_{pq}^{+} + \sum_{p,q=1}^{n}\tilde{A}_{pq}C_{pq}^{-} \\
&+ m\sum_{p,q=1}^{n}(1-\tilde{A}_{pq})\log(1-\tilde{A}_{pq}) + \mathrm{tr}(\tilde{\mathbf{A}}^T\mathbf{D}^{+}\tilde{\mathbf{A}}) - \mathrm{tr}(\tilde{\mathbf{A}}^T\mathbf{D}^{-}\tilde{\mathbf{A}}) \qquad (2)
\end{aligned}
$$

According to the inequality that $z \geq 1 + logz, \forall z > 0$,

$$\tilde{A}_{pq}\log\tilde{A}_{pq} \leq \frac{\tilde{A}_{pq}^2}{A'_{pq}} - \tilde{A}_{pq} + \tilde{A}_{pq}\log A'_{pq} \tag{3}$$

$$(1-\tilde{A}_{pq})\log(1-\tilde{A}_{pq}) \leq \frac{(1-\tilde{A}_{pq})^2}{1-A'_{pq}} - 1 + \tilde{A}_{pq} + (1-\tilde{A}_{pq})\log(1-A'_{pq}) \tag{4}$$

$$\sum_{p,q=1}^{n} \tilde{A}_{pq}C_{pq}^{-} \geq \sum_{p,q=1}^{n} C_{pq}^{-}A'_{pq}\left(1+\log\frac{\tilde{A}_{pq}}{A'_{pq}}\right) \tag{5}$$

$$\mathrm{tr}(\tilde{\mathbf{A}}\mathbf{D}^{-}\tilde{\mathbf{A}}) \geq \sum_{p,q,r=1}^{n} D_{qr}^{-}A'_{qp}A'_{rp}\left(1+\log\frac{\tilde{A}_{qp}\tilde{A}_{rp}}{A'_{qp}A'_{rp}}\right) \tag{6}$$

By the inequality $a \leq \frac{a^2+b^2}{2b}, \forall a, b > 0$,

$$\sum_{p,q=1}^{n} \tilde{A}_{pq}C_{pq}^{+} \leq \sum_{p,q=1}^{n} C_{pq}^{+}\frac{\tilde{A}_{pq}^2 + (A'_{pq})^2}{2A'_{pq}} \tag{7}$$

To handle $tr(\tilde{\mathbf{A}}^T\mathbf{D}^{+}\tilde{\mathbf{A}})$, we introduce the following lemma:

Lemma. *[1] For any nonnegative matrices $\mathbf{X} \in \mathcal{R}^{n\times n}$, $\mathbf{Y} \in \mathcal{R}^{k\times k}$, $\mathbf{Z} \in \mathcal{R}^{n\times k}$, $\mathbf{Z}' \in \mathcal{R}^{n\times k}$, and $\mathbf{X}$, $\mathbf{Y}$ are symmetric, then the following inequality holds*

$$\sum_{i=1}^{n}\sum_{p=1}^{k} \frac{(\mathbf{X}\mathbf{Z}'\mathbf{Y})_{ip}\mathbf{Z}_{ip}^2}{\mathbf{Z}'_{ip}} \geq \mathrm{tr}(\mathbf{Z}^T\mathbf{X}\mathbf{Z}\mathbf{Y})$$

Proof. See [1] □

According to this lemma, we can get

$$\mathrm{tr}(\tilde{\mathbf{A}}^T\mathbf{D}^{+}\tilde{\mathbf{A}}) \leq \sum_{pq=1}^{n} \frac{(\mathbf{D}^{+}\mathbf{A}')_{pq}\tilde{A}_{pq}^2}{A'_{pq}} \tag{8}$$

Take Eq.(3)-Eq.(8) into Eq.(2), we obtain:

$$
\begin{aligned}
&Z(\tilde{\mathbf{A}}, \mathbf{A}') \qquad (9)\\
&= \sum_{p,q=1}^{n} m \left(\frac{\tilde{A}_{pq}^2}{A'_{pq}} + \tilde{A}_{pq} \log A'_{pq} + (1-\tilde{A}_{pq}) \log(1-A'_{pq}) - 1 \right.\\
&\left. + \frac{(1-\tilde{A}_{pq})^2}{1-A'_{pq}} \right) - \sum_{p,q=1}^{n} \left(C_{pq}^{-} A'_{pq} \left(1 + \log \frac{\tilde{A}_{pq}}{A'_{pq}} \right) \right)\\
&+ \sum_{p,q=1}^{n} C_{pq}^{+} \frac{\tilde{A}_{pq}^2 + A_{pq}'^2}{2A'_{pq}} + \sum_{p,q=1}^{n} \frac{(\mathbf{D}^{+}\mathbf{A}')_{pq} \tilde{A}_{pq}^2}{A'_{pq}}\\
&- \sum_{p,q,r=1}^{n} D_{qr}^{-} A'_{qp} A'_{rp} \left(1 + \log \frac{\tilde{A}_{qp} \tilde{A}_{rp}}{A'_{qp} A'_{rp}} \right)\\
&\geq F(\tilde{\mathbf{A}}) \qquad (10)
\end{aligned}
$$

and it is obvious that when $\tilde{\mathbf{A}} = \mathbf{A}'$, $Z(\tilde{\mathbf{A}}, \mathbf{A}') = F(\tilde{\mathbf{A}})$, thus $Z(\tilde{\mathbf{A}}, \mathbf{A}')$ is an auxiliary function of $F(\tilde{\mathbf{A}})$. □

B Proof of Lemma 3

Proof. We now consider the first inequality in Lemma 3. Since $a, b \leq 1$, we have $\sqrt{(\lambda_1+1)^2 - 2\lambda_1(a+b)} \geq \sqrt{(\lambda_1-1)^2} = |\lambda_1 - 1|$. If $\lambda_1 \geq 1$, we have

$$\frac{2\lambda_1 c - \lambda_1 - 1 - \sqrt{(\lambda_1+1)^2 - 2\lambda_1(a+b)}}{2\lambda_1} \leq \frac{2\lambda_1 c - \lambda_1 - 1 - \lambda_1 + 1}{2\lambda_1} = c - 1 \qquad (11)$$

If $0 < \lambda_1 < 1$, then

$$
\begin{aligned}
&\frac{2\lambda_1 c - \lambda_1 - 1 - \sqrt{(\lambda_1+1)^2 - 2\lambda_1(a+b)}}{2\lambda_1} \leq \frac{2\lambda_1 c - \lambda_1 - 1 + \lambda_1 - 1}{2\lambda_1}\\
&= c - \frac{1}{\lambda_1} < c - 1 \qquad (12)
\end{aligned}
$$

To sum up, we have

$$\frac{2\lambda_1 c - \lambda_1 - 1 - \sqrt{(\lambda_1+1)^2 - 2\lambda_1(a+b)}}{2\lambda_1} \leq c - 1 \qquad (13)$$

Then we prove the second inequality in Lemma 3. Since $a, b \geq 0$, we have

$$\frac{2\lambda_1 c - \lambda_1 - 1 + \sqrt{(\lambda_1+1)^2 - 2\lambda_1(a+b)}}{2\lambda_1} \leq \frac{2\lambda_1 c - \lambda_1 - 1 + \lambda_1 + 1}{2\lambda_1} = c \qquad (14)$$

Since $a, b \leq 1$, we have

$$\frac{2\lambda_1 c - \lambda_1 - 1 + \sqrt{(\lambda_1+1)^2 - 2\lambda_1(a+b)}}{2\lambda_1} \geq \frac{2\lambda_1 c - \lambda_1 - 1 + |\lambda_1 - 1|}{2\lambda_1} \tag{15}$$

If $\lambda_1 \geq 1$, then

$$\begin{aligned} &\frac{2\lambda_1 c - \lambda_1 - 1 + \sqrt{(\lambda_1+1)^2 - 2\lambda_1(a+b)}}{2\lambda_1} \geq \frac{2\lambda_1 c - \lambda_1 - 1 + |\lambda_1 - 1|}{2\lambda_1} \\ &= \frac{2\lambda_1 c - \lambda_1 - 1 + \lambda_1 - 1}{2\lambda_1} = c - \frac{1}{\lambda_1} \geq c - 1 \end{aligned} \tag{16}$$

If $\lambda_1 < 1$, then

$$\begin{aligned} &\frac{2\lambda_1 c - \lambda_1 - 1 + \sqrt{(\lambda_1+1)^2 - 2\lambda_1(a+b)}}{2\lambda_1} \geq \frac{2\lambda_1 c - \lambda_1 - 1 + |\lambda_1 - 1|}{2\lambda_1} \\ &= \frac{2\lambda_1 c - \lambda_1 - 1 - \lambda_1 + 1}{2\lambda_1} = c - 1 \end{aligned} \tag{17}$$

To sum up,

$$c - 1 \leq \frac{2\lambda_1 c - \lambda_1 - 1 + \sqrt{(\lambda_1+1)^2 - 2\lambda_1(a+b)}}{2\lambda_1} \leq c$$

□

References

[1] Chris H. Q. Ding, Tao Li, and Michael I. Jordan. Convex and semi-nonnegative matrix factorizations. *IEEE Trans. Pattern Anal. Mach. Intell.*, 32(1):45–55, January 2010.